Stochastic Processes and their Applications 17 (1984) 47–71
North-Holland

LIMIT THEOREMS FOR SUMS DETERMINED BY BRANCHING AND OTHER EXPONENTIALLY GROWING PROCESSES

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Received 27 October 1981
Revised 29 November 1982

A branching process counted by a random characteristic has been defined as a process which at time t is the superposition of individual stochastic processes evaluated at the actual ages of the individuals of a branching population. Now characteristics which may depend not only on age but also on absolute time are considered. For supercritical processes a distributional limit theorem is proved, which implies that classical limit theorems for sums of characteristics evaluated at a fixed age point transfer into limit theorems for branching processes counted by these characteristics. A point is that, though characteristics of different individuals should be independent, the characteristics of an individual may well interplay with the reproduction of the latter. The result requires a sort of L^p-continuity for some $1 \leq p \leq 2$. Its proof turns out to be valid for a wider class of processes than branching ones.

For the case $p = 1$ a number of Poisson type limits follow and for $p = 2$ some normality approximations are concluded. For example results are obtained for processes of rare events, the age of the oldest individual, and the error of population predictions.

This work has been supported by a grant from the Swedish Natural Science Research Council.

AMS 1970 Subject Classification: Primary 60J80; Secondary 60J85

branching process	population
rare event	prediction

1. Introduction

The traditional definition of general branching processes starts from the Ulam–Harris family history space: Let $\mathbb{N}^0 = \{0\}$, $\mathbb{N}$ = the set of natural numbers, and

$$I = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

The interpretation is that $x = (i_1, \ldots, i_n) \in I$ is the i_1th child of ... the i_nth child of the ancestor, who is labelled 0. For details see [6]. With each x there are associated a *reproduction process* ξ_x, i.e. a point process telling at which ages x begets her children $(x, 1)$, $(x, 2)$, etc., and a *characteristic* χ_x, i.e. a random function of age, telling us by which value x is to be counted at different ages. The birth times of individuals are recursively defined by

$$\sigma_0 = 0, \qquad \sigma_{(x,i)} = \sigma_x + \inf\{a;\ \xi_x(a) \geq i\}.$$

Here $\xi_x(a) = \xi_x[0, a]$, the number of children of x up to age a, and inf of the empty set is infinity.

Thus a branching process $\{z_t^\chi\}$ counted by characteristic χ is defined by the superposition of χ's evaluated at the individual's age at t:

$$z_t^\chi = \sum_{x \in I} \chi_x(t - \sigma_x)$$

[6, p. 167]. The characteristics are traditionally assumed to vanish for negative arguments (i.e. no individual is counted before she is born)—we shall refrain from this—and the couples (ξ_x, χ_x), $x \in I$, are usually taken as i.i.d. This may however also be relaxed.

We shall discuss Malthusian, nonlattice, and supercritical processes. This means that the reproduction function

$$\mu(t) = E[\xi(t)]$$

(ξ and χ are used generically for ξ_x and χ_x) is nonlattice, satisfies

$$\mu(0) < 1 < \mu(\infty),$$

and has a Malthusian parameter α defined by

$$\hat{\mu}(\alpha) = \int_0^\infty \mathrm{e}^{-\alpha t} \mu(\mathrm{d}t) = 1$$

(as is certainly the case whenever $\mu(\infty) < \infty$), such that the average age at child-bearing

$$\int_0^\infty t\,\mathrm{e}^{-\alpha t} \mu(\mathrm{d}t) < \infty.$$

For such processes there is a basic growth theorem [8]: As $t \to \infty$

$$\mathrm{e}^{-\alpha t} z_t^\chi \to w\alpha \int_0^\infty \mathrm{e}^{-\alpha t} E[\chi(t)]\,\mathrm{d}t = wE[\hat{\chi}(\alpha)] \quad \text{(in several senses)},$$

for some random variable w under general conditions. Indeed, for convergence in probability it is enough to require (for χ vanishing on the negative axis) that $E[\chi]$ is a.e. continuous (with respect to Lebesgue measure),

$$\sum_{n=0}^\infty \mathrm{e}^{-\alpha n} \sup_{n \leq u < n+1} E[|\chi(u)|] < \infty,$$

and

$$E[\sup_{u \leq t} |\chi(u)|] < \infty \quad \text{for all } t.$$

If further the '$x \log x$'-condition

$$E[\hat{\xi}(\alpha)\log^+\hat{\xi}(\alpha)]<\infty, \quad \hat{\xi}(\alpha)=\int_0^\infty e^{-\alpha t}\xi(dt),$$

holds, the convergence takes place in L^1 and under a slightly different assumption almost surely [8]. Under ('$x \log x$') also $w=0$ if and only if the population ceases to reproduce after some time.

Since the number y_t of individuals born up to time t, grows roughly as $e^{\alpha t}$, by a special case of the theorem (the characteristic $1_{[0,\infty)}$) the general theorem can be viewed as a law of large numbers for summation determined by a population growth. The question arises whether there are corresponding distributional limit theorems for triangular arrays of random characteristics. Indeed, for each t consider characteristics $\{\chi_{tx}\}_{x\in I}$ such that the pairs $\{\xi_x, \chi_{tx}\}_{x\in I}$ are i.i.d.

Under what circumstances will

$$z_t^{\chi_t}=\sum_{x\in I}\chi_{tx}(t-\sigma_x)$$

converge in distribution, as $t\to\infty$?

By the basic growth theorem the case

$$\chi_t(a)=e^{-\alpha t}\chi(a)$$

is already known. Another case has been treated in a thesis by Härnqvist, which deals with the Poisson tendency of streams of either rare occurrences or events in blown up time scales. The latter part can be found in [5] and is not covered by this paper.

2. Main result

The theorem beneath is formulated for branching processes but it holds more generally for (exponentially) growing stochastic processes, see Note 3. It relies upon a uniformity concept, which we name after Lebesgue: A class $\{f_t;\, t\geq 0\}$ of real valued functions of a real variable is *Lebesgue conformable* if (a) for any $T>0$,

$$\sup_{t\geq 0,|u|\leq T}|f_t(u)|<\infty,$$

and (b) for any $\varepsilon>0$ the Lebesgue measure of

$$\left\{u;\ \sup_{|u-u'|<\rho,|u-u''|<\rho}|f_t(u')-f_t(u'')|>\varepsilon\right\}$$

tends to zero uniformly in t, as $\rho\downarrow 0$. If (c) the series

$$\sum_{n=-\infty}^{+\infty} e^{-\alpha n}\sup_{n\leq u\leq n+1}|f_t(u)|$$

converges uniformly, for $\alpha>0$, we shall call the class α-*Wiener*.

These concepts will be applied to various functionals of stochastic processes $\{\chi_{tx}(u);\ u \geq 0\}$, i.i.d. for fixed t, to be used as characteristics. E.g. we shall consider for $\theta \in R$, χ_t generic for χ_{tx},

$$\varphi_{tu}(\theta) = E[e^{i\theta\chi_t(u)}], \qquad h_t(u) = E[\chi_t(u)], \qquad g_t(u) = E[|\chi_t(u)|^p],$$

and

$$D_{t\delta}(u) = \sup_{|v|<\delta} E[|\chi_t(u+v) - \chi_t(u)|^p],$$

for some $1 \leq p \leq 2$, all blown up by a factor $e^{\alpha t}$. For the last of these functions we shall also need an average form of (b).

$$(\bar{\mathrm{b}}) \qquad \lim_{\delta \downarrow 0} \limsup_{t\to\infty} \int_{-\infty}^{+\infty} e^{\alpha t} D_{t\delta}(u)\, e^{-\alpha u}\, du = 0.$$

Theorem. *Consider a nonlattice, supercritical general Malthusian branching process with the Malthusian parameter α, $0<\alpha<\infty$. Assume given, for each $t \geq 0$, a set $\{\chi_{tx}(u)\}$ of stochastic processes (random characteristics) on the line such that $\{\xi_x, \chi_{tx}\}_{x\in I}$ are i.i.d. Denote the characteristic functions, expectations etc. as above and suppose that $\{e^{\alpha t}h_t;\ t \geq 0\}$ and $\{e^{\alpha t}(\varphi_{tu}(\theta)-1);\ t \geq 0\}$, any fixed θ, are Lebesgue conformable and α-Wiener and further that, for some p, $1 \leq p \leq 2$, and all small $\delta > 0$, $g_t(u)$ and $D_{t\delta}(u)$ also define Lebesgue conformable classes $\{e^{\alpha t}g_t\}$ and $\{e^{\alpha t}D_{t\delta}\}$, δ fixed.*

Assume that

$$\lambda = \lim_{t\to\infty} \alpha \int_{-\infty}^{+\infty} e^{\alpha t} h_t(u)\, e^{-\alpha u}\, du \quad \text{and} \quad \hat{\psi}_{\alpha}(\theta) = \lim_{t\to\infty} \int_{-\infty}^{+\infty} e^{\alpha t}\{\varphi_{tu}(\theta)-1\}\, e^{-\alpha u}\, du$$

exist and are finite and that $(\bar{\mathrm{b}})$ holds.

Let w denote the limit in probability of the normed total population, $e^{-\alpha t}y_t \to^{\mathrm{P}} w$. Then

$$z_t^{\chi_t} = \sum_{x\in I} \chi_{tx}(t-\sigma_x)$$

converges in distribution, as $t\to\infty$, to a random variable whose characteristic function is

$$E[e^{w\hat{\psi}_\alpha(\theta)}].$$

If, for all t, $\{\chi_{tx}\}$ is independent of the process $\{y_u;\ u \geq 0\}$ this holds even without the conditions on g_t and $D_{t\delta}$.

Note 1. Condition (b) certainly holds if:

Except possibly for u in a closed Lebesgue null set

$$\lim_{t\to\infty} e^{\alpha t} f_t(u) = f(u)$$

exists, the convergence being uniform over compact sets, and the limit a.e. continuous. Then λ is the Laplace transform

$$\hat{f}(\alpha)=\alpha\int_{-\infty}^{+\infty} e^{-\alpha u}f(u)\,du.$$

Similarly ($\bar{b}$) is a consequence of:

With the same possible exception as above

$$\limsup_{t\to\infty} e^{\alpha t}D_{t\delta}(u)=D_\delta(u)$$

exists, the lim sup being uniform on bounded sets, and $f_\delta \downarrow 0$ a.e. as $\delta\downarrow 0$. These conditions on $\varphi_{tu}(\theta)-1$ are uniform versions of the classical conditions for convergence of sums

$$\sum_{i=1}^{e^{\alpha t}} \chi_{ti}(u).$$

Note 2. If the '$x \log x$'-condition,

$$E[\hat{\xi}(\alpha)\log^+\hat{\xi}(\alpha)]<\infty,$$

is not satisfied, then in the basic growth theorem $w=0$ [5] and the theorem contains little. But it is still true.

Note 3. The theorem has been formulated for branching processes. As will be clear from the proof it holds very generally: Let $\{y'_t\}_{t\geq 0}$ be a nondecreasing, right continuous and nonnegative process such that

$$e^{-\alpha t}y'_t \to \text{some } w',$$

in L^1.

Denote by σ'_n the time of y'_t hitting n,

$$\sigma'_n=\inf\{t\geq 0;\ y'_t\geq n\},$$

and let $\{\chi'_{tn}(u);\ n=1,2,\ldots\}$ be i.i.d. processes on $-\infty<u<+\infty$ satisfying the conditions of the theorem and such that for any k $\{\chi'_{tn};\ n\geq k\}$ is independent of $\sigma'_1,\ldots,\sigma'_k,\chi_{t1},\ldots,\chi_{tk-1}$. Then

$$\sum_n \chi'_{tn}(t-\sigma'_n)$$

converges in distribution and the limit has the characteristic function given.

The branching process $z_t^{\chi_t}$ can be written in this form by ordering the individuals in the order they were born (or suitably (cf. [8]) for simultaneously born individuals) and letting σ'_n denote the birth time of the nth individual and χ'_{tn} its characteristic. The independence of $\{\chi'_{tn};\ n\geq k\}$ and $\sigma'_i,\ i\leq k,\ \chi'_{ti},\ i\leq k-1$, follows from the independence of (χ_{tx},ξ_x) pertaining to different individuals.

For branching processes we can also handle the case where $\{e^{-\alpha t}y_t\}$ is not uniformly integrable. Then [4], $e^{-\alpha t}y_t \to^{P} 0$ and we shall show directly that so does $z_t^{\chi_t}$.

3. Proofs.

The Theorem will be proved by a sequence of lemmas, which are given for the general case of Note 3. Thus, let $\{y'_t;\, t \geq 0\}$ be any nondecreasing, nonnegative stochastic process, $w' \geq 0$ a random variable and $\alpha > 0$ a real number such that

(i) $\quad e^{-\alpha t}y'_t \xrightarrow{P} w'$,

and

(ii) $\quad \sup_t e^{-\alpha t}E[y'_t] < \infty$.

Define

$$\sigma'_n = \inf\{t \geq 0;\ y_t \geq n\}$$

and let, for each t, χ'_{tn}, $n = 1, 2, \ldots$, be i.i.d. stochastic proccsses such that, for each k, $\{\chi'_{tn}\}_{n>k}$ and $\sigma'_1, \ldots, \sigma'_k$ are independent. This will be assumed throughout in the following. For simplicity we also take y_t as integer valued.

Lemma 1. *If $\{e^{\alpha t}f_t\}$ is Lebesgue conformable and α-Wiener, and $\{y'_t\}$ is as above, then*

$$\sum_n f_t(t - \sigma'_n) \xrightarrow{P} w'\lambda,$$

provided

$$\lambda = \lim_{t\to\infty} \int_{-\infty}^{\infty} \alpha\, e^{\alpha t} f_t(u)\, e^{-\alpha u}\, du$$

exists.

Proof. Let $\varepsilon > 0$ be given. For any integer $T > 0$

$$P\left[\left|\sum_k f_t(t - \sigma'_k) - w'\lambda\right| > \varepsilon\right]$$

$$\leq P\left[\left|\sum_{|t-\sigma'_k| \geq T} f_t(t - \sigma'_k)\right| > \tfrac{1}{2}\varepsilon\right] + P\left[\left|\sum_{|t-\sigma'_k| < T} f_t(t - \sigma'_k) - w'\lambda\right| > \tfrac{1}{2}\varepsilon\right].$$

But

$$\sum_{|t-\sigma'_k| \geq T} |f_t(t - \sigma'_k)| \leq \sum_{|n| \geq T}\ \sum_{n \leq t - \sigma'_k < n+1} e^{-\alpha(t-\sigma'_k)} e^{\alpha t} |f_t(t - \sigma'_k)|\, e^{-\alpha\sigma'_k}$$

$$\leq \sum_{|n| \geq T} e^{-\alpha n} \sup_{n \leq u < n+1} e^{\alpha t}|f_t(u)| \sum_{t-n-1 < \sigma'_k \leq t-n} e^{-\alpha\sigma'_k}$$

$$\leq \sum_{|n| \geq T} e^{-\alpha n} \sup_{n \leq u < n+1} e^{\alpha t}|f_t(u)|\, e^{-\alpha(t-n)} y'_{t-n}\, e^{\alpha}.$$

Markov's inequality yields that

$$P\left[\sum_{|t-\sigma'_k|>T} |f_t(t-\sigma'_k)| > \tfrac{1}{2}\varepsilon\right] \leq (2e^{\alpha}/\varepsilon) \sum_{|n|\geq T} e^{-\alpha n} \sup_{n\leq u<n+1} e^{\alpha t}|f_t(u)| \sup_v e^{-\alpha v} E[y'_v].$$

By (c) and the assumption that

$$\sup_{t\geq 0} e^{-\alpha t} E[y_t] < \infty,$$

$T >$ some T_ε renders this less than ε. We fix such a T which is also large enough to guarantee that

$$\alpha \sup_{t\geq 0} \sum_{|n|\geq T} e^{-\alpha n} \sup_{n\leq u<n+1} e^{\alpha t}|f_t(u)| < \varepsilon/2c,$$

where c has been chosen so that also

$$P(w' > c) < \varepsilon.$$

Then we define

$$\nu_t(u) = \begin{cases} 0 & u < -T, \\ e^{-\alpha t}(y'_{t+T} - y'_{t-u}), & -T \leq u \leq T, \\ e^{-\alpha t}(y'_{t+T} - y'_{t-T}), & u > T. \end{cases}$$

Clearly,

$$\sum_{|t-\sigma'_k|\leq T} f_t(t-\sigma'_k) = \int e^{\alpha t} f_t(u)\nu_t(du).$$

Now, let $\{t_n\}$ be any sequence tending to infinity. By using first the characterization of convergence in probability through a.s. convergent subsequences [3, p. 67] and then the diagonal argument, we can produce a subsequence $\{t_{n'}\}$ such that a.s.

$$\nu_{t_{n'}}(u) \to w'(e^{\alpha T} - e^{-\alpha u})$$

for all $u \in [-T, T]$. The Lebesgue conformability implies that $\{e^{\alpha t} f_t\}$ is a Lebesgue uniformity class on $[-T, T]$ in the terminology of [1]. Therefore [1, theorem 1]

$$\limsup_{t'_n\to\infty} \left| \int e^{-\alpha t_{n'}} f_{t_{n'}}(u)\nu_{t_{n'}}(du) - \lambda w' \right|$$

$$\leq \limsup_{t'_n\to\infty} \left| \int e^{-\alpha t_{n'}} f_{t_{n'}}(u)\nu_{t_{n'}}(du) - \int_{-T}^{T} \alpha\, e^{\alpha(t_{n'}-u)} f_{t_{n'}}(u) w'\, du \right|$$

$$+ w'\left(\limsup_{t\to\infty} \int_{|u|>T} \alpha\, e^{-\alpha(t-u)} |f_t(u)|\, du \right)$$

$$\leq w'\alpha \sup_t \sum_{|n|\geq T} e^{-\alpha n} \sup_{n\leq u<n+1} e^{\alpha t}|f_t(u)| \leq w'\varepsilon/2c.$$

Hence from an arbitrary sequence $\{t_n\}$ tending to infinity we can extract a subsequence $t_{n'}$ such that a.s.

$$\lim_{t_{n'}\to\infty} \sum_{|t_{n'}-\sigma'_k|\le T} f_{t_{n'}}(t_{n'}-\sigma'_k)$$

exists and lies between $w'(\lambda-\varepsilon/2c)$ and $w'(\lambda+\varepsilon/2c)$. It follows that

$$P\left[\left|\sum_{|t_n-\sigma'_k|\le T} f_{t_n}(t_n-\sigma'_k)-w'\lambda\right|>\tfrac{1}{2}\varepsilon\right]$$

$$\le P\left[\left|\sum_{|t_n-\sigma'_k|\le T} f_{t_n}(t_n-\sigma'_k)\right|>\tfrac{1}{2}\varepsilon,\ w'=0\right]$$

$$+P\left[\left|\sum_{|t_n-\sigma'_k|\le T} f_{t_n}(t_n-\sigma'_k)-w'\lambda\right|>\varepsilon w'/2c, 0<w'\le c\right]+P[w'>c]$$

can only exceed 2ε for finitely many $t_{n'}$. Since $\{t_n\}$ was an arbitrary sequence, this implies that

$$\limsup_{t\to\infty} P\left[\left|\sum_{|t-\sigma'_k|\le T} f_t(t-\sigma'_k)-w'\lambda\right|>\tfrac{1}{2}\varepsilon\right]\le 2\varepsilon,$$

and the convergence in probability has been proved. □

Note. For nonstochastic y'_t and continuous f_t of the form $e^{-\alpha t}f$ this is a case of Wiener's general Tauberian theorem [10, p. 214].

Lemma 2. *With $\{f_t\}$ as in Lemma 1 but the assumption $e^{-\alpha t}y'_t \xrightarrow{P} w'$ replaced by*

$$e^{-\alpha t}E[y'_t]\to \textit{some } \gamma<+\infty,$$

the following holds:

$$E\left[\sum_n f_t(t-\sigma'_n)\right]\to\gamma\lambda.$$

Proof. In the preceding proof we must only replace $\nu_t(u)$ by $E[\nu_t(u)]$. The subsequence arguments simplify since (for $|u|\le T$)

$$E[\nu_t(u)]\to\gamma(e^{\alpha T}-e^{-\alpha u}).\ \square$$

Lemma 3. *If $\{e^{\alpha t}f_{t\delta}\}$ satisfies* (a), ($\bar{\text{b}}$) *and* (c), *and*

$$e^{-\alpha t}E[y'_t]\to \textit{some } \gamma,$$

then

$$\lim_{\delta\downarrow 0}\limsup_{t\to\infty} E\left[\sum_n f_{t\delta}(t-\sigma'_n)\right]=0.$$

Proof. Again patterned after that of Lemma 1. □

Lemma 1 is enough to prove our theorem for $\{\chi'_{tn}\}$ independent of $\{y'_t\}$. Indeed,

$$\sum_n \{\varphi_{t,t-\sigma'_n}(\theta)-1\} \xrightarrow{\mathrm{P}} w'\hat{\psi}_\alpha(\theta).$$

Conditions (a) and (c) imply that

$$\sup_u |\varphi_{t,u}(\theta)-1| \to 0 \quad \text{as } t\to\infty,$$

and, as above, that

$$\limsup_{t\to\infty} E\left[\sum_n |\varphi_{t,t-\sigma'_n}(\theta)-1|\right]$$

$$\leqslant E\left[\sum_k \sum_{k\leqslant t+\sigma_{n'}<k+1} \mathrm{e}^{-\alpha k} \sup_{k\leqslant u<k+1} \mathrm{e}^{\alpha t}|\varphi_{t,u}(\theta)-1|\,\mathrm{e}^{-\alpha\sigma'_n}\right]$$

$$\leqslant \mathrm{e}^{\alpha} \sup_t E[\mathrm{e}^{-\alpha t}y'_t] \sum_k \mathrm{e}^{-\alpha k} \sup_{k\leqslant u<k+1} \mathrm{e}^{\alpha t}|\varphi_{t,u}(\theta)-1|$$

is bounded in t, (a) taking care of the central part of the sum and (c) of the tail.

From the complex inequality

$$|\log(1+z)-z| \leqslant |z|^2, \quad |z|\leqslant \tfrac{1}{2},$$

we then deduce that, for t large enough,

$$E\left[\left|\sum_n \log \varphi_{t,t-\sigma'_n}(\theta) - \sum_n (\varphi_{t,t-\sigma'_n}(\theta)-1)\right|\right] \leqslant E\left[\sum_n |\varphi_{t,t-\sigma'_n}(\theta)-1|^2\right]$$

$$\leqslant \sup_u |\varphi_{t,u}(\theta)-1|\, E\left[\sum_n |\varphi_{t,t-\sigma_n}(\theta)-1|\right].$$

Thus

$$\left|\sum_n \log \varphi_{t,t-\sigma'_n}(\theta) - \sum_n (\varphi_{t,t-\sigma'_n}(\theta)-1)\right| \xrightarrow{\mathrm{P}} 0,$$

as $t\to\infty$, and consequently

$$\sum_n \log \varphi_{t,t-\sigma'_n}(\theta) \xrightarrow{\mathrm{P}} w'\hat{\psi}_\alpha(\theta),$$

and

$$E\left[\prod_n \varphi_{t,t-\sigma'_n}(\theta)\right] \to E[\mathrm{e}^{w'\hat{\psi}_\alpha(\theta)}],$$

by dominated convergence. But if $\{\chi'_{tn}\}$ and $\{y'_t\}$ are independent,

$$E[\mathrm{e}^{\mathrm{i}\theta\sum_n \chi'_{tn}(t-\sigma'_n)}] = E\left[\prod_n \varphi_{t,t-\sigma'_n}(\theta)\right].$$

In the general case $\{\sigma_n'\}$ will be approximated by a simpler sequence—this is where the L^p-continuity assumption ($\bar{\mathrm{b}}$) enters. The clue is given by the following

Lemma 4. *As* $n \to \infty$,

$$\sigma_n' - (1/\alpha)\log n \xrightarrow{\mathrm{P}} -(1/\alpha)\log w' \qquad (=+\infty \quad \text{if } w'=0).$$

Proof. Since

$$\sigma_n' - (1/\alpha)\log n \leqslant u \quad \Leftrightarrow \quad y'_{u+(1/\alpha)\log n} \geqslant n,$$

the assumed convergence of $\mathrm{e}^{-\alpha t} y_t'$ implies that the indicators

$$1_{\{\sigma_n' - (1/\alpha)\log n \leqslant u\}} = 1_{\{\mathrm{e}^{-\alpha(u+(1/\alpha)\log n)} y'_{u+(1/\alpha)\log n} \geqslant \mathrm{e}^{-\alpha u}\}} \xrightarrow{\mathrm{P}} 1_{\{-(1/\alpha)\log w' \leqslant u\}},$$

in all points of continuity of the distribution function of $(1/\alpha)\log w'$. The proof will be complete after (the simple but useful)

Lemma 5. *If* $Y, Y_1, Y_2, \ldots$ *are random variables, then* $Y_n \to^{\mathrm{P}} Y$ *if and only if*

$$1_{\{Y_n \leqslant t\}} \xrightarrow{\mathrm{P}} 1_{\{Y \leqslant t\}},$$

for all t *such that* $P[Y = t] = 0$.

Proof. It is trivial that $Y_n \to^{\mathrm{P}} Y$ yields the convergence of indicators. For the converse let $\{n_k\}$ be any subsequence of the natural numbers. By the subsequence characterization [3, p. 67] and diagonal argument again we can find a subsequence $\{n_k'\}$ such that

$$1_{\{Y_{n_k'} \leqslant t\}} \xrightarrow{\text{a.s.}} 1_{\{Y \leqslant t\}},$$

for all t in a countable dense set without Y-mass and hence for all t with $P[Y = t] = 0$. Therefore $Y_{n_k'} \to^{\text{a.s.}} Y$ and convergence in probability follows from the converse of the subsequence characterization. $\square$

Proof of the theorem. Lemma 4 makes it natural to approximate σ_n' by

$$\hat{\sigma}_n = (1/\alpha)(\log n - \log w').$$

However this need not be independent of $\{\chi_m'\}$. Therefore we use a two step procedure: For fixed k,

$$\sigma_n^k = (1/\alpha)\{\log n - \log(k\,\mathrm{e}^{-\alpha\sigma_k'})\}$$

is determined by σ_k' and is therefore independent (by assumption) of $\{\chi_m';\ n \geqslant k\}$.

Still it should be close to $\hat{\sigma}_n$, and therefore to σ'_n, at least for large k, since $k\,\mathrm{e}^{-\alpha\sigma_k} \to^{\mathrm{P}} w'$ by Lemma 4.

For any random variable $\tilde{w} \geq 0$ a process $\tilde{y}_t$ is defined by

$$\tilde{y}_t = \#\{n; (1/\alpha)(\log n - \log \tilde{w}) \leq t\} = [\tilde{w}\,\mathrm{e}^{\alpha t}].$$

If $E[\tilde{w}] < \infty$, then clearly

$$\mathrm{e}^{-\alpha t}\tilde{y}_t \xrightarrow{\mathrm{P}} \tilde{w},$$

and

$$\mathrm{e}^{-\alpha t}E[\tilde{y}_t] \to E[\tilde{w}].$$

Hence, the conclusions of Lemmas 1 and 2 must hold (which can also be proved directly by Riemann sums). For the choice $\tilde{w} = w_k = k\,\mathrm{e}^{-\alpha\sigma'_k}$

$$\sum_{n=1}^{k-1} \chi'_{tn}(t - \sigma_n^k) \xrightarrow{\mathrm{P}} 0$$

as $t \to \infty$ by (a) and (c). However (abusing the notation)

$$\sum_{n \geq k} \chi'_{tn}(t - \sigma_n^k) \xrightarrow{\mathrm{d}} E[\mathrm{e}^{w_k \hat{\psi}_\alpha(\theta)}],$$

by the proved case of the theorem (y'_t and χ'_{tn} independent). It follows that for any k

$$\sum_n \chi'_{tn}(t - \sigma_n^k) \xrightarrow{\mathrm{d}} E[\mathrm{e}^{w_k \hat{\psi}_\alpha(\theta)}],$$

as $t \to \infty$.

Since $w_k \xrightarrow{\mathrm{P}} w'$ and

$$|\mathrm{e}^{w_k \hat{\psi}_\alpha(\theta)}| \leq 1$$

as a characteristic function (to the power w_k), dominated convergence yields that

$$\lim_{k \to \infty} E[\mathrm{e}^{w_k \hat{\psi}_\alpha(\theta)}] = E[\mathrm{e}^{w' \hat{\psi}_\alpha(\theta)}].$$

To check that

$$\sum_n \chi'_{tn}(t - \sigma'_n) \quad (t \to \infty)$$

has this same limit in distribution as

$$\sum_n \chi'_{tn}(t - \sigma_n^k) \quad (\text{first } t \to \infty, \text{ then } k \to \infty)$$

we consider

$$P\left[\left|\sum_n \chi'_{tn}(t - \sigma'_n) - \chi'_{tn}(t - \sigma_n^k)\right| > \varepsilon\right]$$

for an arbitrary $\varepsilon > 0$. This cannot exceed

$$P\left[\left|\sum_{n=1}^{k-1} \chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)\right| > \tfrac{1}{2}\varepsilon\right] + P\left[\left|\sum_{n\geq k} \chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)\right| > \tfrac{1}{2}\varepsilon\right].$$

By choosing $t >$ some T_k we can make the first term $< \varepsilon$. To the second one we apply first Markov's inequality and then the generalized Minkowski inequality due to von Bahr, Esseen, and in this generality Chatterji [2] (which states that for some A_p, dependent only upon p, $1 \leq p \leq 2$,

$$E\left[\left|\sum_1^m X_n\right|^p\right] \leq A_p \sum_1^m E[|X_n|^p]$$

holds (trivially and) always for $p = 1$ and if the X_j are martingale differences for $1 < p \leq 2$ [3, p. 391]). In our case we write

$$\mathscr{B}_n = \sigma(\sigma'_1, \ldots, \sigma'_{n+1}, \chi'_{t1}, \ldots, \chi'_{tn}).$$

Then, for $n > k$, $\chi'_{tn}(t-\sigma'_n)$ and $\chi'_{tn}(t-\sigma^k_n)$ are both measurable with respect to $\mathscr{B}_n$ and χ'_{tn} is independent of $\mathscr{B}_{n-1}$ whereas σ'_n and σ^k_n both belong to the latter. By the assumptions made on

$$h_t(u) = E[\chi_t(u)]$$

in the theorem,

$$\sum_n h_t(t-\sigma'_n) \xrightarrow{\mathrm{P}} w'\lambda, \qquad \sum_n h_t(t-\sigma^k_n) \xrightarrow{\mathrm{P}} w'_k\lambda,$$

where

$$\lambda = \lim_{t\to\infty} \alpha \int_{-\infty}^{+\infty} e^{\alpha t} h_t(u)\, e^{-\alpha u}\, du.$$

Therefore we may as well assume that

$$E[\chi'_{tn}(u)] = 0,$$

and hence that $\{\chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n);\, n \geq k\}$ are martingale differences with respect to $\{\mathscr{B}_n;\, n \geq k\}$.

Thus by Markov and Minkowski–Chatterji and, for any $\delta > 0$,

$$\begin{aligned}
&P\left[\left|\sum_{n\geq k} \chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)\right| > \tfrac{1}{2}\varepsilon\right] \\
&\quad\leq (2/\varepsilon)^p E\left[\left|\sum_{n\geq k} \chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)\right|^p\right] \\
&\quad\leq (2/\varepsilon)^p A_p \sum_{n\geq k} E[|\chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)|^p] \\
&\quad= (2/\varepsilon)^p A_p \left\{\sum_{n\geq k} E[|\chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)|^p;\, |\sigma'_n-\sigma^k_n| < \delta]\right.
\end{aligned}$$

$$+\sum_{n\geq k} E[|\chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)|^p;|\sigma'_n-\sigma^k_n|\geq\delta]\Big\}$$

$$=(2/\varepsilon)^pA_p\Big\{\sum_{n\geq k} E[E[|\chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)|^p|\mathscr{B}_{n-1}];|\sigma'_n-\sigma^k_n]<\delta]$$

$$+\sum_{n\geq k} E[E[|\chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma^k_n)|^p|\mathscr{B}_{n-1}];|\sigma'_n-\sigma^k_n|\geq\delta]\Big\}$$

$$\leq(2/\varepsilon)^pA_p\Big\{E\Big[\sum_{n\geq k}\sup_{|v|<\delta} E[|\chi'_{tn}(t-\sigma'_n)-\chi'_{tn}(t-\sigma'_n+v)|^p|\mathscr{B}_{n-1}]\Big]$$

$$+E\Big[\sum_{n\geq k}2E[|\chi'_{tn}(t-\sigma'_n)|^p+|\chi'_{tn}(t-\sigma^k_n)|^p|\mathscr{B}_{n-1}];|\sigma'_n-\sigma^k_n|\geq\delta\Big]\Big\}.$$

In the notation of the Theorem,

$$D_{t\delta}(u)=\sup_{|v|<\delta} E[|\chi_t(u)+\chi_t(u+v)|^p],$$

the first term is

$$E\Big[\sum_{n\geq k} D_{t\delta}(t-\sigma'_n)\Big]\leq E\Big[\sum_{n\geq 1} D_{t\delta}(t-\sigma'_n)\Big].$$

By assumption and Lemma 3 the lim sup as $t\to\infty$ of this tends to zero as $\delta\downarrow 0$. As to the second term, we consider first

$$\sum_{n\geq k} E[E[|\chi'_{tn}(t-\sigma'_n)|^p|\mathscr{B}_{n-1}];|\sigma'_n-\sigma^k_n|\geq\delta]=E\Big[\sum_{j=-\infty}^{+\infty}\sum_{\substack{j\leq t-\sigma'_n<j+1\\ n\geq k}} g_t(t-\sigma'_n)\gamma_n\Big],$$

where

$$g_t(u)=E[|\chi_t(u)|^p],$$

and

$$\gamma_n=1_{\{|\sigma'_n-\sigma^k_n|\geq\delta\}}.$$

Then

$$\sum_j\sum_{\substack{j\leq t-\sigma_n<j+1\\ n\geq k}} g_t(t-\sigma'_n)\gamma_n\leq\sum_j \mathrm{e}^{-\alpha j}\sup_{j\leq u<j+1}\mathrm{e}^{\alpha t}g_t(u)\,\mathrm{e}^{-\alpha(t-j)}\sum_{n=k}^{y'_{t-j}}\gamma_n\,.$$

Since $0\leq\gamma_n\leq 1$,

$$\mathrm{e}^{-\alpha t}\sum_{n=k}^{y'_t}\gamma_n\leq\mathrm{e}^{-\alpha t}\sum_{n=k}^{C\mathrm{e}^{\alpha t}}\gamma_n+\mathrm{e}^{-\alpha t}y'_t1_{\{y'_t>C\mathrm{e}^{\alpha t}\}}$$

for any $C>0$. Hence

$$E\Big[\mathrm{e}^{-\alpha t}\sum_{n=k}^{y'_t}\gamma_n\Big]\leq E[\mathrm{e}^{-\alpha t}y'_t;\,w'=0]+C\sup_{n\geq k}P[|\sigma'_n-\sigma^k_n|>\delta;\,w'>0]$$

$$+E[\mathrm{e}^{-\alpha t}y'_t;\mathrm{e}^{-\alpha t}y'_t>C].$$

The first term tends to zero and the last term can be made little for all t by choice of a large C, exactly when $\{e^{-\alpha t}y_t'\}$ is uniformly integrable. For branching processes this occurs under the '$x \log x$'-condition,

$$E[\hat{\xi}(\alpha)\log^+\hat{\xi}(\alpha)]<\infty$$

[4]. In this case we must therefore for $\varepsilon>0$ choose first C, then δ and then k to make $\hat{D}_{t\delta}(\alpha)$ and $C\sup_{n\geq k}P[|\sigma_n'-\sigma_n^k|>\delta,\ w'>0]$, respectively, sufficiently small, and finally let $t\to\infty$. This completes the proof of the theorem under '$x \log x$' or, generally, uniform integrability.

We turn to branching processes with

$$E[\hat{\xi}(\alpha)\log^+\hat{\xi}(\alpha)]=\infty.$$

As mentioned, then $e^{-\alpha t}y_t\to^{P}0$ [4]. Let again $\varepsilon>0$ be given. For any integer T and $\delta>0$, we have, writing

$$g_t(u)=E[|\chi_t(u)|^p],$$

$$P\left[\left|\sum_n \chi_{tn}'(t-\sigma_n')\right|>\varepsilon\right]\leq P\left[\sum_{|t-\sigma_n'|>T}|\chi_{tn}'(t-\sigma_n')|>\tfrac12\varepsilon\right]$$

$$+P\left[\sum_{|t-\sigma_n|\leq T}|\chi_{tn}'(t-\sigma_n')|>\tfrac12\varepsilon\right]$$

$$\leq A_p(2/\varepsilon)^pE\left[\sum_{|t-\sigma_n'|>T}|\chi_{tn}'(t-\sigma_n')|^p\right]+P[y_{t+T}>\delta\, e^{\alpha(t+T)}]$$

$$+A_p(2/\varepsilon)^pE\left[\sum_{\substack{|t-\sigma_n'|\leq T\\ 1\leq n\leq\delta e^{\alpha(t+T)}}}|\chi_{tn}'(t-\sigma_n')|^p\right]$$

$$\leq A_p(2/\varepsilon)^p\sum_{|k|\geq T}E\left[\sum_{k\leq t-\sigma_n'<k+1}e^{\alpha t}g_t(t-\sigma_n')\,e^{-\alpha(t-\sigma_n')}\,e^{-\alpha\sigma_n'}\right]$$

$$+P[y_{t+T}>\delta\, e^{\alpha(t+T)}]+(2/\varepsilon)^pA_pE\left[\sum_{\substack{|t-\sigma_n'|\leq T\\ 1\leq n\leq\delta e^{\alpha(t+T)}}}g_t(t-\sigma_n')\right]$$

$$\leq(2/\varepsilon)^pA_p\sum_{|k|\geq T}e^{-\alpha k}\sup_{k\leq|u|<k+1}e^{\alpha t}g_t(u)\,e^{-\alpha(t-k)}E[y_{t-k}]\,e^{\alpha}$$

$$+P[y_{t+T}>\delta\, e^{\alpha(t+T)}]+(2/\varepsilon)^pA_p\sup_{|u|\leq T}e^{\alpha t}g_t(u)\delta\, e^{\alpha T}.$$

Here we can first choose T to make the first sum less than $\frac13\varepsilon$ and then find δ so that so are the two second terms for $t\geq$ some $t(\delta,T)$. Here we used $\sup_t e^{-\alpha t}E[y_t]<\infty$, the uniform convergence in (c), $e^{-\alpha t}y_t\to^{P}0$, and (a).

The theorem is now proved. □

4. Applications

A. Rare events

Consider events $\{E_t^x, t \geq 0\}$, which may occur in an individual x's life with a little probability $p_t \sim \lambda\, e^{-\alpha t}$ as $t \to \infty$. Assume that the individual's age at the occurrence—if it takes place—has the distribution G_t, and that G_t converges weakly to G as $t \to \infty$. If $\chi_t(u)$ just tells whether the event has occurred or not up to age u, then

$$\varphi_{tu}(\theta) = p_t G_t(u)\, e^{i\theta} + 1 - p_t G_t(u).$$

Hence

$$e^{\alpha t}\{\varphi_{tu}(\theta) - 1\} \to \lambda G(u)(e^{i\theta} - 1)$$

in continuity points of G. Thanks to the monotonicity of G_t, (b) holds and since all G_t vanish on the negative axis $\varphi_{tu}(\theta) - 1$ also satisfies (c). Also

$$h_t(u) = E[\chi_t(u)] = E[|\chi_t(u)|] = p_t G_t(u)$$

meets the three conditions.

Finally, for $p = 1$,

$$D_{t\delta}(u) = p_t \max\{G_t(u+\delta) - G_t(u), G_t(u) - G_t(u-\delta)\}$$

so that

$$e^{\alpha t} D_{t\delta}(u) \to \lambda \max\{G(u+\delta) - G(u), G(u) - G(u-\delta)\}$$

for u, $u+\delta$ and $u-\delta$ outside some countable set, and (a), (c) and the joint part of (b) and ($\bar{\text{b}}$) are satisfied for the relevant functions. Further

$$\limsup_{t\to\infty} \int_{-\infty}^{+\infty} e^{\alpha t} D_{t\delta}(u)\, e^{-\alpha u}\, du \leq \lambda \int_0^{\infty} (G(u+\delta) - G(u-\delta))\, e^{-\alpha u}\, du,$$

which certainly tends to zero as $\delta \downarrow 0$. Thus, by the theorem the number $z_t^{\chi_t}$ of such E_t-events up to time t is asymptotically as $t \to \infty$, mixed Poisson with the parameter $\lambda w \hat{G}(\alpha)$.

With a similar argument it can be shown that if $I_1, \ldots, I_n$ are disjoint intervals with finite right endpoints and

$$\chi_t(u) = \begin{cases} 1 & \text{if the event occurs at an age in the set } u + \bigcup_{j=1}^{n} I_j, \\ 0 & \text{otherwise} \end{cases}$$

(here $\chi_t(u)$ need not vanish for u negative!), then $z_t^{\chi_t}$, the number of occurrences in the set $t + \bigcup_{j=1}^n I_j$, converges in distributions towards a mixed Poisson variable with the parameter

$$\lambda w \hat{G}(\alpha) \alpha \int_{\bigcup_{1\leq j\leq n} I_j} e^{\alpha u}\, du$$

and thus we can conclude the weak convergence (with respect to the semi-weak topology induced by convergence of the integrals of continuous bounded functions with a support bounded to the right) of the point processes

$$\eta_t(v) = \text{number of occurrences before } t+v$$

towards a mixed Poisson process [7, Theorem 4.7].

B. Rare events at evanescing or oscillating ages

Consider events $\{E_t, t \geq 0\}$, E_t occurring at an age with the (subprobability) distribution G_t. Suppose that with a_t defined, for some $\lambda > 0$, as

$$a_t = \log\{\lambda / \hat{G}_t(\alpha)\} - t,$$

the integrals

$$\alpha \int_{-\infty}^{+\infty} e^{\alpha t} G_t(a_t + u)\, e^{-\alpha u}\, du = \lambda$$

are uniformly convergent (in t). The process $z_t^{\chi_t}$,

$$\chi_t(u) = \begin{cases} 1 & \text{if } E_t \text{ has occurred before age } u + a_t, \\ 0 & \text{otherwise,} \end{cases}$$

then counts the number of E_t-occurrences before $t + a_t$. The monotonicity of G_t and the uniform convergence assumption implies that $h_t(u) = G_t(a_t + u)$ satisfies (a), the first part of (b) and (c). The second part of (b) follows from the definition of a_t. Similarly, $\varphi_{tu}(\theta) - 1$ satisfies all conditions. Finally, with $p = 1$, $D_{t\delta}$ satisfies the joint part (b) and ($\bar{\text{b}}$), along with h_t, and certainly

$$\lim_{\delta \downarrow 0} \limsup_{t \to \infty} \int_{-\infty}^{+\infty} e^{\alpha t} D_{t\delta}(u)\, e^{-\alpha u}\, du = 0.$$

Hence we can conclude that the number $z_t^{\chi_t}$ of E_t-occurrences before $t + a_t$ is asymptotically, as $t \to \infty$, mixed Poisson with parameter λw.

Just as in A, the weak convergence of the point process $\eta_t(v) =$ number of occurrences before $t + a_t + v$ towards a mixed Poisson process can be shown.

As an illustration we consider the process of *hero mothers*. Assume that, for each $n = 1, 2, \ldots$, $P[\xi(\infty) > n] > 0$, whereas $P[\xi(\infty) < \infty] = 1$. Let E_t be the event of ever obtaining n_t children for some $n_t \to \infty$. Then

$$G_t(u) = P[\xi(u) \geq n_t],$$

and if the a_t-translated Laplace transform of this converges uniformly in t, condition (c) must hold. (A sufficient condition for the existence of such $\{a_t\}$ is that

$$\limsup_{n \to \infty} P[\xi(a) \geq n \mid \xi(\infty) > n] > 0$$

for some a.) We conclude that the number of hero mothers is, approximately, mixed Poisson.

A second illustration is provided by the *attainment of high ages*. Here we suppose that individuals have a finite life following a distribution function L, which should be left continuous at the right end point of its support (possibly $=+\infty$). Then we can find s_t such that $L(s_t)<1$ but $L(s_t)\uparrow 1$. With $E_t=\{\text{celebration of the } s_t\text{th birthday}\}$ this fits into the present framework.

Define

$$a_t=\log(\lambda/\{1-L(s_t)\}(1-e^{-\alpha s_t})).$$

Then the integrals

$$\alpha\int_{s_t-a_t}^{+\infty} e^{\alpha t}(1-L(s_t))1_{\{a_t+u\geq s_t\}}\,e^{-\alpha u}\,du$$

$$=\alpha\int_{s_t-a_t}^{+\infty}(e^{\alpha t}(1-L(s_t))\,e^{\alpha(s_t-a_t)})\,e^{-\alpha(u-(s_t-a_t))}\,ds=\lambda$$

are uniformly convergent, since $e^{\alpha t}(1-L(s_t))\,e^{-\alpha(s_t-a_t)}\leq\lambda$.

Hence the appropriate mixed Poisson convergence follows for the number of s_t birthdays.

C. *The old individuals*

Assume that life spans follow a proper distribution function L with $L(u)<1$ for $u<+\infty$. For t not too small there exists an a_t with the property

$$e^{\alpha(t+a_t)}\alpha\int_{a_t}^{+\infty}\{1-L(u)\}\,e^{-\alpha u}\,du=1.$$

Generally we let a_t be any numbers such that the left hand side converges to one, as $t\to\infty$. Similarly, to any such $\{a_t\}$ and $v>0$ we let $u_t(v)$ be such that

$$\lim_{t\to\infty} e^{\alpha(t+a_t)}\alpha\int_{u_t(v)}^{+\infty}\{1-L(u)\}\,e^{-\alpha u}\,du=v.$$

Also this can always be chosen so that equality really holds from some t_v onwards. Clearly

$$\lim_{t\to\infty}\frac{\int_{u_t(v)}^{+\infty}\{1-L(u)\}\,e^{-\alpha u}\,du}{\int_{a_t}^{+\infty}\{1-L(u)\}\,e^{-\alpha u}\,du}=v.$$

But, for any real numbers $a<b$,

$$\int_a^{+\infty}\{1-L(u)\}\,e^{-\alpha u}\,du\geq e^{\alpha(b-a)}\int_b^{+\infty}\{1-L(u)\}\,e^{-\alpha u}\,du.$$

Hence, if $v' > v$, we must have that

$$e^{\alpha|u_t(v)-a_t|} \leq v',$$

if only $t >$ some t'.

We now define the characteristic $\chi_t(u)$, which is one if the individual is alive at $t+a_t$ with $u > u_t(v) - a_t$ and zero otherwise. This yields a variable $z_t^{\chi_t}$ counting the number of living individuals older that $u_t(v)$ at time not t but $t+a_t$.

In order to satisfy conditions (a) and (b) we require that

$$\inf_{u \geq 0} \{1 - L(u+1)\}/\{1 - L(u)\} = a > 0,$$

(as is true if $1 - L(u)$ has an exponential tail). Then, for $t > t'$,

$$v' \geq e^{\alpha(t+a_t)}\alpha \int_{u_t(v)}^{+\infty} \{1 - L(u)\} e^{-\alpha u} du$$

$$\geq e^{\alpha(t+a_t)}\alpha \int_{u_t(v)}^{u_t(v)+1} \{1 - L(u)\} e^{-\alpha u} du$$

$$\geq \alpha\, e^{\alpha(t+a_t)}\{1 - L(u_t(v))\} a\, e^{-\alpha\{u_t(v)+1\}},$$

which implies that

$$\sup_{-\infty < u < +\infty} e^{\alpha t} E[\chi_t(u)] = \sup_{u > u_t(v) - a_t} e^{\alpha t}\{1 - L(u + a_t)\}$$

$$= e^{\alpha t}\{1 - L(u_t(v))\} < v'\, e^{|u_t(v) - a_t|}\, e^{\alpha}/\alpha a \leq v'^2\, e^{\alpha}/\alpha a$$

if only $t > t'$. Together with the fact that $\chi_t(u)$ vansihes for $u < -|u_t(v) - a_t|$, i.e. certainly for $u < -|\log v'|/\alpha$ and $t > t'$, this means that $\varphi_{tu}(\theta) - 1$ and $h_t(u) = E[\chi_t(u)]$ must satisfy (a) and (c), at least for $t > t'$. The first part of (b) follows from distribution function properties (recall the form of $\varphi_{tu}(\theta) - 1$, e.g.). The second part of (b) follows from the requirements on a_t and $u_t(v)$, like

$$\alpha \int_{-\infty}^{+\infty} e^{\alpha t} h_t(u)\, e^{-\alpha u} du = \alpha \int_{-\infty}^{+\infty} e^{\alpha t}\{1 - L(u + a_t)\} e^{-\alpha u} du$$

$$= e^{\alpha(t+a_t)}\alpha \int_{u_t(v)}^{+\infty} \{1 - L(u)\} e^{-\alpha u} du \to v.$$

The conditions on $D_{t\delta}$ and g_t for $p = 1$ are easily checked. It follows that the number $z_t^{\chi_t}$ of individuals older than $u_t(v)$ at $t + a_t$ is, asymptotically as $t \to \infty$, mixed Poisson with the parameter vw.

As an application of this we note that the *age U_t of the oldest individual* at $t + a_t$ satisfies

$$P[U_t \leq u_t(v)] = P[z_t^{\chi_t} = 0] \to E[e^{-vw}].$$

To make this more explicit consider the case of

$$1-L(u)\sim\gamma\,\mathrm{e}^{-\beta u}$$

as $u\to\infty$. Then

$$\int_a^{+\infty}\{1-L(u)\}\,\mathrm{e}^{-\alpha u}\,\mathrm{d}u\sim\gamma\,\mathrm{e}^{-(\alpha+\beta)a}/(\alpha+\beta)$$

as $a\to\infty$. Hence we can choose $a_t=\alpha t/\beta$ and

$$u_t(v)=\alpha t/\beta+(\alpha+\beta)^{-1}\log\{\alpha\gamma/v(\alpha+\beta)\}$$

to obtain

$$P[U_t-\alpha t/\beta\leq(\alpha+\beta)^{-1}\log\{\alpha\gamma/v(\alpha+\beta)\}]\to E[\mathrm{e}^{-vw}],$$

or in terms of the maximal age at t, $A_t=U_{\beta t/(\alpha+\beta)}$,

$$P[A_t-\alpha t/(\alpha+\beta)\leq s]\to E[\exp\{-\alpha\gamma w\,\mathrm{e}^{-(\alpha+\beta)s}/(\alpha+\beta)\}],$$

as $t\to\infty$.

Returning to the general case, we note that if u_t is defined by equality, it will not increase in v. Hence we can consider nonincreasing u_t, a sequence of intervals defined by $l_1\leq r_1\leq l_2\leq r_2\leq\cdots\leq l_n\leq r_n$, and let $\chi_t(u)=1$ exactly when

$$u_t(r_j)<u+a_t\leq u_t(l_j)$$

for some $j=1,2,\ldots,n$ and the individual is alive at time $u+a_t$. In the same manner as above it can be shown that $z_t^{\chi_t}$ converges to a mixed Poisson variable, this time with parameter

$$w\sum_{j=1}^{n}(r_j-l_j).$$

Hence, as in A the weak convergence as $t\to\infty$ follows of the process $\eta_t(v)$, counting the number of individuals older than $u_t(v)$ at time $t+a_t$.

D. The random error of population projection type predictions

Classical population projections are predictions of some future property of the population, which take into account the present age distribution of living individuals, but no other stochastic properties. This means that they have, λ_x being x's life span, the form

$$\sum_x f(t-\sigma_x)1_{\{\lambda_x>t-\sigma_x\}},$$

for various functions f like:

(i) the expected size of the progeny of a u-aged individual a time units hence,

$$f(u)=E[z_a(u)]/P[\lambda>u],$$

$z_a(u)$ denoting a process started from a u-aged ancestor at time zero [6, p. 176], or

(ii) the expected number of children to be borne by those alive now,

$$f(u)=E[\xi(u,\infty)]/P[\lambda>u]=\{m-\mu(u)\}/P[\lambda>u],$$

$m=\mu(\infty)$ being the expected number of children per individual, or

(iii) the expected remaining life over some (retiring) age a, $\vee$ denoting maximum,

$$f(u)=E[(\lambda-u\vee a)^+]/P[\lambda>u].$$

In the last two cases it is easy to construct characteristics that measure the random error of the predictions made,

$$\chi_t(u)=e^{-\alpha t/2}\{\xi(u,\infty)-(m-\mu(u))1_{\{\lambda>u\}}/P[\lambda>u]\}$$

and

$$\chi_t'(u)=e^{-\alpha t/2}\{(\lambda-u\vee a)^+-E[(\lambda-u\vee a)^+]1_{\{\lambda>u\}}/P[\lambda>u]\}$$

respectively. Then

$$h_t(u)=E[\chi_t(u)]=E[\chi_t'(u)]=0,$$

and by the central limit theorem

$$e^{\alpha t}\{\varphi_{tu}(\theta)-1\}\to\tfrac{1}{2}\theta^2V(u)$$

where φ_{tu} is the characteristic function in question and $V(u)$ denotes $P[\lambda>u]$ times the conditional variance of $\xi(u,\infty)$ or $(\lambda-u\vee a)^+$, given $\lambda>u$. The summed random variables are dominated,

$$e^{\alpha t}\chi_t^2(u)\leq\xi^2(\infty)+g(u),\qquad e^{\alpha t}\chi_t'^2(u)\leq\lambda^2+g'(u),$$

where g and g' are bounded (except possibly in a neighbourhood of the right end point of the support of the life span). Hence if the second moments of $\xi(\infty)$ or λ are finite, the convergences can be seen to be uniform (outside the neighbourhoods mentioned) by appropriate Taylor expansions. Also

$$V(u)\leq\mathrm{Var}[\xi(u,\infty)]\leq E[\xi^2(\infty)]\quad\text{and}\quad V(u)\leq E[\lambda^2],$$

respectively. Hence V is bounded, and also of bounded variation (cf the next example). This guarantees the conditions (a), (b) and (c) of the Theorem for the case $p=2$. Neither is condition ($\bar{\text{b}}$) difficult to verify.

Hence

$$z_t^{\chi_t}\xrightarrow{d}N(0,w\hat{V}(\alpha)),$$

and similarly for $z_t^{\chi_t'}$, in an obvious notation for mixed normal random variables.

The population size prediction problem in (i) is more subtle, since the progeny $z_a(u)$ of a u-aged individual is not a random characteristic. However, we can construct a true random characteristic

$$\{z'_{a\chi}(u);\ u\geq 0\}$$

such that the conditional distributions of the processes $z'_{ax}(u)$ and $z_{ax}(u)$ given (ξ_x, λ_x) coincide (i.e. $\{\xi_x, \lambda_x, \{z'_{ax}(u);\ u \geq 0\}\}_{x \in I}$ are i.i.d.). then

$$z_{t+a} \overset{d}{=} \sum_{\sigma_x \leq t} z'_{ax}(t-\sigma_x)$$

and hence we can still apply our theorem, but to

$$e^{-\alpha/2}\{z'_a(u) - E[z_a(u)]1_{\{\lambda > u\}}/P[\lambda > u]\}.$$

For the verification of conditions it is enough to assume that

$$E\left[\left\{\int_0^\infty u\, e^{-\alpha u}\xi(du)\right\}^2\right] < \infty,$$

as is certainly the case whenever $\xi(\infty)$ has a finite second moment. This will be developed further in the next section.

E. Approximation of the limit variable w

Consider a projection type prediction of

$$w = \lim_{t\to\infty} e^{-\alpha t} y_t \quad \text{(in probability)}, \qquad \hat{w}_t = \sum_{\sigma_x \leq t} v(t-\sigma_x) 1_{\{\lambda_x > t-\sigma_x\}},$$

where

$$v(u) = \lim_{t\to\infty} E[e^{-\alpha t} y_t(u)]/P[\lambda > u]$$

$$= e^{\alpha u} \int_{u^+}^{\infty} e^{-\alpha a}\mu(da) \Big/ \left(P[\lambda > u] \int_0^\infty a\, e^{-\alpha a}\mu(da)\right)$$

is Fisher's reproductive value of a u-aged individual [6, p. 212] divided by the average age at childbearing,

$$\beta = \int_0^\infty a\, e^{-\alpha a}\mu(da).$$

and $y_t(u)$, like $z_t(u)$ above, is one plus the total progenies up to time $t+u$ of the daughter processes initiated by the ancestor after attaining age u. The interpretation is that this is a process started at time zero by a u-aged ancestor, the probability law not being conditioned upon this ancestor surviving age u. In the notation

$$w_x(a) = \lim_{t\to\infty} e^{-\alpha t} y_{tx}(a) \quad \text{in probability}$$

the limit variable w can, for any $t \geq 0$, to be expressed as

$$w = \sum_{\sigma_x \leq t} e^{-\alpha t} w_x(t-\sigma_x).$$

As in D(i) the $w_x(u)$ are not random characteristics but again true characteristics

$w'_x(u)$ can be constructed so that

$$w \stackrel{d}{=} \sum_{\sigma_x \le t} e^{-\alpha t} w'_x(t-\sigma_x)$$

and the theorem can be tried on

$$\chi_{t,x}(u) = e^{-\alpha t/2}\{w'_x(u) - v(u)1_{\{\lambda_x > u\}}\}.$$

We must then assume that $E[w^2] < \infty$, which is the case as soon as $E[\hat{\xi}^2(\alpha)] < \infty$. Then also

$$v(u) = E[w(u)].$$

By Taylor expansions the conditions on the characteristic functions of $\chi_t(u)$, needed for this, are reduced to corresponding requirements on

$$e^{\alpha t}\,\mathrm{Var}[\chi_t(u)] = E[\mathrm{Var}[w(u)\,|\,1_{\{\lambda > u\}}]] = P[\lambda > u]\,\mathrm{Var}[w(u)\,|\,\lambda > u].$$

Condition (a) can be directly verified. For condition (b) an elementary regression formula yields

$$\begin{aligned} E[\mathrm{Var}[w(u)\,|\,1_{\{\lambda > u\}}]] &= \mathrm{Var}[w(u)] - \mathrm{Var}[E[w(u)\,|\,1_{\{\lambda > u\}}]] \\ &= E[w^2(u)] - E^2[w(u)] + P[\lambda > u]P[\lambda \le u]v^2(u). \end{aligned}$$

But splitting w according to the individuals of the first generation, we obtain

$$w(u) = e^{\alpha u} \sum_{\sigma_{(i)} > u} e^{-\alpha\sigma_{(i)}} w_{(i)}(0).$$

Conditionally upon ξ_0 the $w_{(i)}(0)$ appearing here are i.i.d. This means that $E[w^2(u)]$ and $E^2[w(u)]$ must both have bounded variation on finite intervals. The same must be true for the last term. This yields (b) on any finite intervals. Outside finite intervals we shall take recourse to (c):

By the regression formula and the splitting above (we make the assumption $w(0) = w$, i.e. $\xi(0) = 0$, to simplify notation)

$$\begin{aligned} e^{\alpha t}\,\mathrm{Var}[\chi_t(u)] &\le \mathrm{Var}[w(u)] \\ &= E\left[e^{2\alpha u} \sum_{\sigma_{(i)} > u} e^{-2\alpha\sigma_{(i)}}\right] \mathrm{Var}[w] + \mathrm{Var}\left[e^{\alpha u} \sum_{\sigma_{(i)} > u} e^{-\alpha\sigma_{(i)}}\right] E^2[w] \\ &= e^{2\alpha u} \int_{u^+}^{\infty} e^{-2\alpha a}\mu(da)\,\mathrm{Var}[w] + \mathrm{Var}\left[e^{\alpha u} \int_{u^+}^{\infty} e^{-\alpha a}\xi(du)\right] E^2[w]. \end{aligned}$$

Replacing the summation condition (c) by integration, which is possible in the present case, we see that the condition is valid if

$$\int_0^{\infty} e^{-\alpha u}\left\{e^{2\alpha u} \int_{u^+}^{\infty} e^{-2\alpha a}\mu(da)\right\} du = \int_0^{\infty} e^{\alpha u}\left\{\int_{u^+}^{\infty} e^{-2\alpha}\mu(da)\right\} du \le \hat{\mu}(\alpha)/\alpha < \infty$$

(which is true by assumption) and if

$$\int_0^\infty e^{-\alpha u}\,\mathrm{Var}\left[e^{\alpha u}\int_{u^+}^\infty e^{-\alpha a}\xi(\mathrm{d}a)\right]\mathrm{d}u=\int_0^\infty \mathrm{Var}\left[e^{\alpha u/2}\int_{u^+}^\infty e^{-\alpha a}\xi(\mathrm{d}a)\right]\mathrm{d}u<\infty,$$

which is the case under, e.g., the *assumption* that

$$E\left[\left\{\int_0^\infty u\,e^{-\alpha u/2}\xi(\mathrm{d}u)\right\}^2\right]<\infty.$$

Indeed,

$$\begin{aligned}\mathrm{Var}\left[e^{\alpha u/2}\int_{u^+}^\infty e^{-\alpha a}\xi(\mathrm{d}a)\right]&\leqslant E\left[\left\{u\int_u^\infty e^{-\alpha a/2}\xi(\mathrm{d}a)\right\}^2\right]\Big/u^2\\&\leqslant E\left[\left\{\int_0^\infty a\,e^{-\alpha a/2}\xi(\mathrm{d}a)\right\}^2\right]\Big/u^2.\end{aligned}$$

Certainly other monotone square integrable functions that $1/u$ would do. The asymptotic continuity condition ($\bar{\mathrm{b}}$) follows in the same manner as above for $p=2$. Hence

$$e^{-\alpha t/2}(\hat{w}_t-w)\xrightarrow{\mathrm{d}}N(0,w\hat{V}(\alpha)),$$

where

$$V(u)=\mathrm{Var}[w(u)\,|\,\lambda>u]P[\lambda>u].$$

5. A remark on the lattice case

Consider a branching process, whose reproduction is of a lattice type. Then a discrete version of our main result holds. As a matter of fact conditions simplify, the continuity requirements (part of (b) and all of ($\bar{\mathrm{b}}$)) becoming irrelevant, thanks to the essential discreteness of time and age.

For notational simplicity assume that reproduction can take place only at ages 0, 1, 2,... and consider the process only at time points $t=0, 1, 2, \ldots$. Then $h_t(u)$ and $\varphi_{tu}(\theta)-1$ should satisfy, t and n taking only integer values,

($\hat{\mathrm{a}}$) $\quad \sup_{t,|n|\leqslant T} e^{\alpha t}|f_t(n)|<\infty,$

($\hat{\mathrm{b}}$) $\quad \lambda=\lim_{t\to\infty}(1-e^{-\alpha})\sum_{n=-\infty}^{+\infty}e^{\alpha t}f_t(n)\,e^{-\alpha n},$

exists and is finite,

($\hat{\mathrm{c}}$) $\quad$ and the sum is uniformly absolutely convergent.

The function $\hat{\psi}_\alpha(\theta)$, determining the limit, turns into the sum

$$\lim_{t\to\infty}(1-\mathrm{e}^{-\alpha})\sum_{n=-\infty}^{+\infty}\mathrm{e}^{\alpha t}\{\varphi_{tn}(\theta)-1\}\,\mathrm{e}^{-\alpha n}.$$

Lemma 1 follows from the convergence

$$\nu_t(\{n\})\xrightarrow{\mathrm{P}} w(1-\mathrm{e}^{-\alpha})\,\mathrm{e}^{-\alpha n},$$

for $n=-T,-T+1,\ldots,T$ and $t\to\infty$ on the integers.

The first difficulty arises with Lemma 4, which now takes the form

$$\sigma'_n-[(1/\alpha)\log(n/w)]\xrightarrow{\mathrm{P}}1,\quad w>0,$$

$$\sigma'_n-(1/\alpha)\log n\to\infty,\qquad w=0.$$

Here w is the limit in probability, as $n\to\infty$, of $\mathrm{e}^{-\alpha n}y_n$, existing by a lattice version of [8, Theorem 3.1] and square brackets denote integer part. The proof produces from an arbitrary given sequence a subsequence $\{n'\}$, such that

$$(1/\alpha)\log n'-[(1/\alpha)\log n']\to\text{some } c.$$

Since w is continuous on $(0,\infty)$ [9], it follows that

$$[(1/\alpha)\log n'/w]-[(1/\alpha)\log n']\to[c-(1/\alpha)\log w]$$

a.s. on $w>0$. But that

$$\sigma'_{n'}-[(1/\alpha)\log n']\xrightarrow{\mathrm{P}}[c-(1/\alpha)\log w]+1$$

follows as in Lemma 4.

For the further approximation we use, instead of $w_k=k\,\mathrm{e}^{-\alpha\sigma'_k}$, the martingale [8]

$$R_k=1+\sum_{n=0}^{k}\mathrm{e}^{-\alpha\sigma'_n}\{\hat{\xi}'_n(\alpha)-1\}$$

with a.s. limit $\alpha\beta w$,

$$\beta=\int_0^\infty t\,\mathrm{e}^{-\alpha t}\mu(\mathrm{d}t)$$

as before. With the corresponding

$$\sigma^k_n=[(1/\alpha)\log(n\alpha\beta/R_k)]+1$$

it holds, by the continuity of w, that

$$\sigma'_n-\sigma^k_n=\sigma'_n-[(1/\alpha)\log(n/w)]-1+[(1/\alpha)\log(n/w)]-[(1/\alpha)\log(n\alpha\beta/R_k)]\xrightarrow{\mathrm{P}}0,$$

on $w > 0$, as first n and then $k > \infty$. The rest of the proof runs smoothly with $\delta < 1$, since then $|\sigma'_n - \sigma^n_k| < \delta$ implies that the two actually coincide.

Our applications hence generalize to the lattice case, where the rate of convergence results (D) and (E) might be of some interest. In the case of Galton–Watson processes the predictor $\hat{w}_n$ of w in (E) is proportional to normed nth generation itself, and, thus, a theorem by Heyde on the rate of convergence of supercritical Galton–Watson processes [6, p. 38], follows.

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